Proof Complexity of Topological Coloring Principles

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[SAT2014] + [ICALP2015] (with James Aisenberg, Maria Luisa Bonet, Sam Buss)

Outline

Preliminaries: proofs, complexity, topology, coloring

[SAT2014] $Kneser_{k,n}$: Lower bounds on proof complexity

[SAT2014] $Kneser_{k,n}$: Efficient proofs for $k = 2$ and $k = 3$

[ICALP2015] $Kneser_{k,n}$: Efficient proofs for the general case

[ICALP2015] *Tucker*: New candidate problem based on topological principles

Conclusions and Future Work

Propositional Frege systems

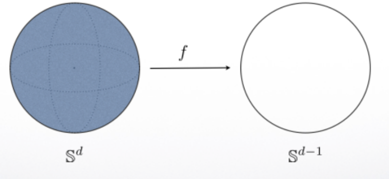
- Frege - *textbook* proof systems,
  - finite schema of axioms ($A \wedge B \rightarrow A$) and inference rules ($A, A \rightarrow B \vdash B$),
  - proof: list of axioms or formulas obtained from previous formulas in the proof by inference rules,
  - *size* of a proof: number of symbols,
  - intuitively, reason using (Boolean) formulas,
  - different Frege systems poly-equivalent.
- Extended Frege - allow renaming.
  - intuitively, reason using Boolean circuits.

Proofs and complexity

- (conjecture:) Boolean formulas require exponential size to simulate Boolean circuits,
- (by analogy:) is there an exponential separation between Frege and Extended Frege proofs?
- candidates?
- early: *PHP* poly-size Extended Frege proofs (induction); poly-size Frege proofs via an efficient counting argument (Buss),
- Istrate, Crăciun: is $Kneser_{k,n}$ such a candidate (for what k)?

Algebraic Topology: Trials and Tribulations

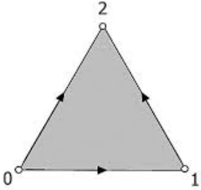
- Borsuk-Ulam: cannot map continuously and antipodally *n*-dim. sphere into a sphere of lower dimension.

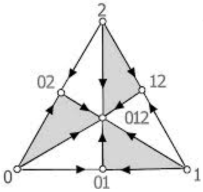


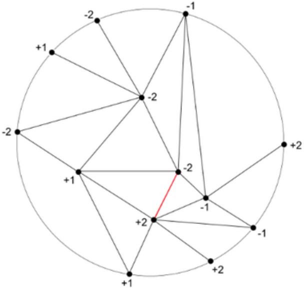
- Dimensionality = global obstruction.
- Obstruction = unsat?

Algebraic Topology: Trials and Tribulations (cont.)

- Tucker's lemma (cont.):







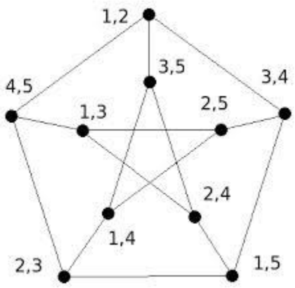
- Tucker's lemma: unsat exponential-size propositional formula, *$Kneser_{k,n}$* by substitution.
- exponential large objects,
- get around: low dimensional Tucker lemma (low dimensional skeleton)
- still: exponential objects :((no poly Frege/Extended Frege).
- Kneser via Tucker is not obviously Frege/Extended Frege.

Kneser's Conjecture

- Stated in 1955 (Martin Kneser, Jaresbericht DMV)
- Let $n \geq 2k - 1 \geq 1$. Let $c : \binom{n}{k} \to [n - 2k + 1]$. Then there exist two disjoint sets A and B with $c(A) = c(B)$.
- $k = 1$ PHP!
- $k = 2, 3$ combinatorial proofs (Stahl, Garey & Johnson)
- $k \geq 4$: only proved in 1977 (Lovász) using algebraic topology.
- Combinatorial proofs known (Matousek, Ziegler). "hide" Alg. Topology
- No "purely combinatorial" proof known

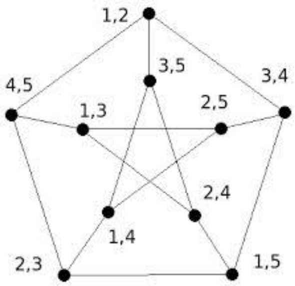
Kneser's Conjecture (cont.)

- the chromatic number of a certain graph $Kn_{n,k}$ (at least) $n - 2k + 2$. (exact value)
- Vertices: $\binom{n}{k}$. Edges: disjoint sets.
- E.g. $k = 2$, $n = 5$: Petersen's graph has chromatic number (at least) three.



Kneser's conjecture: stronger form - Schrijver's Theorem

- inner cycle in Petersen's graph already chromatic number three.
- $A \in \binom{n}{k}$ stable if it doesn't contain consecutive elements i, $i+1$ (including $n, 1$).
- Schrijver's Theorem: Kneser's conjecture holds when restricted to stable sets only.



$\mathit{Kneser}_{k,n}$ as a Propositional Formula

- naïve encoding $X_{A,k} = \mathit{TRUE}$ iff A colored with color k.
- $X_{A,1} \vee X_{A,2} \vee \ldots \vee X_{A,n-2k+1}$ "every set is colored with (at least) one color"
- $\overline{X_{A,j}} \vee \overline{X_{B,j}}$ $(A \cap B = \emptyset)$ "no two disjoint sets are colored with the same color"
- Fixed k: $\mathit{Kneser}_{k,n}$ has poly-size (in n).
- Extends encoding of PHP

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Conclusions and Future Work

Reduction from $Kneser_{k+1,n}$ to $Kneser_{k,n-2}$

- There exists a variable substitution $\Phi_k : Var(Kneser_{k+1,n}) \to Var(Kneser_{k,n-2})$ s.t. $\Phi_k(Kneser_{k+1,n})$ consists precisely of the clauses of $Kneser_{k,n-2}$ (perhaps repeated and in a different order)
- Let $A \in \binom{n}{k+1}$. Define $\Phi_k(X_{A,i})$ by:
  - **Case 1:** $A_{\leq k} \subseteq [n-2]$**:** $\Phi_k(X_{A,i}) = Y_{A_{\leq k},i}$
  - **Case 2:** $A_{\leq k} \not\subseteq [n-2]$**:** $(n-1, n \in A)$
    Let $A = P \cup \{n-1, n\}$, $|P| = k-1$. Let $\lambda = max\{j : j \leq n-2, j \notin P\}$. Define $\Phi_k(X_{A,i}) = Y_{P \cup \{\lambda\},i}$
- Clause $X_{A,1} \vee X_{A,2} \vee \ldots \vee X_{A,n-2k+1}$ maps to $Y_{B,1} \vee Y_{B,2} \vee \ldots \vee Y_{B,n-2k+1}$, $B = A$ (Case 1).
- Clauses $\overline{X_{A,i}} \vee \overline{X_{B,i}}$ $(A \cap B = \emptyset)$ map to $\overline{Y_{C,i}} \vee \overline{Y_{D,i}}$
- Case 2 cannot happen for both A and B. By case analysis $C \cap D = \emptyset$.
- Schrijver: similar, but check that mapping preserves stability.

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$k = 2$ (semantic) proof

- Basic result: Given any $(n-3)$-coloring c of $\binom{n}{2}$ and color $1 \leq l \leq n-3$, at least one of the following alternatives is true:

  1. there exist two disjoint sets $D, E \in c^{-1}(l)$.
  2. $|c^{-1}(l)| \leq 3$.
  3. there exists $x \in [n]$, $x \in \bigcap_{A \in c^{-1}(l)} A$.

  i.e. the color classes:

  - either have at most 3 elements,
  - or they have a common element, which we call special.
- Immediately yields poly-size Extended Frege proof.

$k = 2$ (semantic) proof (cont.)

- Now we count:
  - $p_r = |\{1 \leq \lambda \leq r : |c^{-1}(\lambda)| \geq 4 \text{ and } \bigcap_{A \in c^{-1}(\lambda)} A \neq \emptyset\}|$,
    i.e. the number of classes with special elements.
  - $s_r = |\{i \in [n] : \bigcap_{A \in c^{-1}(\lambda)} A = \{i\} \text{ for some } 1 \leq \lambda \leq r \text{ with } |c^{-1}(\lambda)| \geq 4\}|$,
    the number of special elements.
  - $M_r = \sum_{i=1}^{r} |c^{-1}(i)|$,
    the number of elements colored by the first r colors
  - $N_r = p_r(n-1) - \frac{p_r(p_r-1)}{2} + 3(r - p_r)$
    first the number of elements from color classes with at least one special element, and then the rest.
- Then prove:
  - Sequences M_r, N_r are monotonically increasing.
  - For $1 \leq r \leq n-3,\ M_r \leq N_r$.
  - $N_{n-3} \leq \binom{n}{2} - 3$ which establishes the contradiction (we have to cover $\binom{n}{2}$ sets).

$k = 2$, propositional simulation

- Buss counting formulas: binary encodings of the number of variables set to TRUE.
- can be simulated by Frege proofs.
- Examples:
  - For $n \geq 5$ and $1 \leq l \leq n-3$,

$$\bigvee_{\substack{D,E \in \binom{n}{2} \\ D \cap E = \emptyset}} (X_{D,l} \wedge X_{E,l}) \vee [Count((X_{S,l})_{S \in \binom{n}{2}}) \leq 3] \vee \bigvee_{i \in [n]} (\bigwedge_{i \notin S} \overline{X_{S,l}}).$$

  -

$$M_r = |\{A \in \binom{n}{2} : \bigvee_{1 \leq l \leq r} X_{A,l}\}| \ \big(\text{semantically } = \sum_{i=1}^{r} |c^{-1}(i)|\big),$$

$$M_r^{(1)} = |\{A \in \binom{n}{2} : \bigvee_{1 \leq l \leq r} (X_{A,l} \wedge [Count(X_{S,l}) \leq 3]) \}|,$$

$$M_r^{(2)} = |\{A \in \binom{n}{2} : \bigvee_{1 \leq l \leq r} (X_{A,l} \wedge [Count((X_{S,l})_{S \in \binom{n}{2}}) \geq 4]) \}|$$

$k = 3$, polynomial *EF*

- For any $1 \leq \lambda \leq n-5$ at least one of the following is true:
  1. $c^{-1}(\lambda)$ contains two disjoint sets
  2. $|c^{-1}(\lambda)| \leq 3n-8$, or
  3. there exists $x \in \cap_{A \in c^{-1}(\lambda)} A$.
- Try counting:
  1. Count p_r, number of sets $c^{-1}(l)$, $1 \leq l \leq t$ such that $|c^{-1}(\lambda)| \geq 3n-7$ (implicitly $\cap_{A \in c^{-1}(l)} A \neq \emptyset$).
  2. Define $M_r^{(3)} = \sum_{i=1}^{r} |c^{-1}(i)|$ and
  $$N_r^{(3)} = \binom{n-1}{2} + \binom{n-2}{2} + \ldots + \binom{n-p_r}{2} + (n-5-p_r)(3n-7).$$
  3. Show that $M_r^{(3)} \leq N_r^{(3)}$.
  4. Obtain a contradiction from $M_{n-5}^{(3)} = \binom{n}{3}$ and $N_{n-3}^{(3)} < \binom{n}{3}$.
- No longer possible to establish the contradiction.
- Extended Frege proof (successive elimination of elements and color classes).

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Conclusions and Future Work

Efficient proofs for the general case

Theorem Let k be fixed. The Kneser-Lovász Theorem has *polynomial size* extended Frege proofs, and *quasipolynomial* size Frege proofs.

- New proof based on counting.
- Idea:
  - Color class P_l: *star-shaped* (nonempty intersection), or non-starshaped.
  - Finitely many base cases, up to an $N(k)$ - solve directly.

Efficient proofs for the general case (cont'd)

- Polynomial Extended Frege:

Lemma: Fix $k > 1$. There is an $N(k)$ so that, for $n > N(k)$, any $(n - 2k + 1)$-coloring of $\binom{n}{k}$ has at least one star-shaped color class.

  - By infinite descend, delete one color and central element, therefore reduce the size of the problem.
  - $n - N(k)$ rounds ($N(k) = k^4$),
  - this can be translated in poly Extended Frege.

- Quasipolynomial Frege:

Lemma: Fix $k > 1$ and $0 < \beta < 1$. Then there exists an $N(k, \beta)$ such that for $n > N(k, \beta)$, if c is an $(n - 2k + 1)$-coloring of $\binom{n}{k}$, then c has at least $\frac{n}{k}\beta$ many star-shaped color classes.

  - By infinite descend, delete $1/2k$ of the colors in one step, until $n < 2k^2(k - 1/2)$.
  - $O(logn)$ rounds, giving quasipolynomial Frege.

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Octahedral Tucker Lemma

- Octahedral ball: $\mathcal{B}^n := \{(A, B) : A, B \subseteq [n] \text{ and } A \cap B = \emptyset\}$.
- Antipodal mapping: $\lambda : \mathcal{B}^n \to \{1, \pm 2, \ldots, \pm n\}$ is *antipodal* if $\lambda(\emptyset, \emptyset) = 1$, and for all other pairs $(A, B) \in \mathcal{B}^n$, $\lambda(A, B) = -\lambda(B, A)$.
- Complementarity: (A_1, B_1) and (A_2, B_2) in $\mathcal{B}^n$ are *complementary* w.r.t. an antipodal map λ on $\mathcal{B}^n$ if $A_1 \subseteq A_2$, $B_1 \subseteq B_2$ and $\lambda(A_1, B_1) = -\lambda(A_2, B_2)$.
- Tucker lemma: If $\lambda : \mathcal{B}^n \to \{1, \pm 2, \ldots, \pm n\}$ is antipodal, then there are two elements in $\mathcal{B}^n$ that are complementary.

Truncated Tucker Lemma

- Octahedral ball: $\mathcal{B}^n := \{(A, B) : A, B \subseteq [n] \text{ and } A \cap B = \emptyset\}$.
- Antipodal mapping: $\lambda : \mathcal{B}^n \to \{1, \pm 2, \ldots, \pm n\}$ is *antipodal* if $\lambda(\emptyset, \emptyset) = 1$, and for all other pairs $(A, B) \in \mathcal{B}^n$, $\lambda(A, B) = -\lambda(B, A)$.
- Complementarity
  - $A_1 \preceq A_2$ iff $(A_1 \cup A_2)_{\leq k} = A_2$.
  - $(A_1, B_1) \preceq (A_2, B_2)$ iff $A_1 \preceq A_2$, $B_1 \preceq B_2$, and $A_i \cap B_j = \emptyset$ for $i, j \in \{1, 2\}$.
  - (A_1, B_1) and (A_2, B_2) are *k-complementary* w.r.t. an antipodal map λ on $\mathcal{B}^n_k$ if $(A_1, B_1) \preceq (A_2, B_2)$ and $\lambda(A_1, B_1) = -\lambda(A_2, B_2)$.
- Tucker lemma: Let $n \geq 2k > 1$. If $\lambda : \mathcal{B}^n_k \to \{\pm 2k \ldots, \pm n\}$ is antipodal, then there are two elements in $\mathcal{B}^n_k$ that are k-complementary.

Tucker $\longrightarrow$ *Truncated Tucker* $\longrightarrow$ Kneser

- Octahedral Tucker: exponential size (no short proofs).
- Truncated Tucker: poly size translations (but how about proofs?)
- Kneser: can be obtained from Truncated Tucker, therefore
  - weaker,
  - *new candidate* for separation of Frege and Extended Frege!
- Kneser: $2 - DTucker$ (also $k - Tucker$) PPA-complete [Buss2015] (not PPAD as previously known - Papadimitriou)

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Further work

- Other problems: cutting necklace, etc.
- Other proof systems: e.g. cutting planes (k=2), polynomial calculus, etc.
- Logics for implicit proof systems?
- Bounded reverse mathematics
- Topological obstructions: from graph coloring to CSP.
- Topological arguments as sound (but incomplete) implicit proof systems
  - if $K \not\to L$ then a "proof of $A \not\to B$" is a pair of embeddings $(K \to A), (B \to L)$.
  - Checking soundness ($K \not\to L$) may not be polynomial. If K, L "standard objects" we could omit proof of $K \not\to L$ from complexity

Thank you!

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