Worst-case fairness in TU-cooperative games

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- Using rational assessments, homo economicus attempts to maximize utility:**B accepts**
- Any other cost sharing $A : x$, $B : 27 - x$ with $9 \leq x \leq 18$ mutually advantageous, **not necesary equitable**

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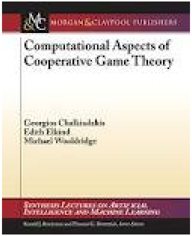
- Rational allocation

$$\forall S \subseteq N : \sum_{i \in S} x_i \leq c(S).$$

Cooperative games "in practice"



- Common investments (roads, highways, ISP networks, ...).
- Nodes in sensors network. Distributed source coding. Information theory (Slepian-Wolf inequalities) $\Rightarrow$ cooperative game
- European union voting.
- Sociology ("power in exchange networks")
- Economy: strategic models for social networks (Jackson-Wolinsky).
- Artificial Intelligence: multiagent systems
- . . . ...

Cooperative game: The CORE

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- Is not empty when the cost function is **submodular:**

$$(S \subseteq T) \rightarrow c(T \cup \{x\}) - c(T) \leq c(S \cup \{x\}) - c(S).$$

- This is a **concave** game.

Shapley Value

$$S(i) = \frac{1}{n!} \cdot \left[\sum_{\sigma \in S_n} X_i^{(\sigma)} \right]$$

- is the gravity center of the n! dots from core
- is the fair solution
- is in the core, for concave game.
- $1/1$ pizza: 13.5 lei each.

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- Incompatible with other social norms e.g. egalitarianism Dutta& Ray (Econometrica 1989)
- No alternative to the Shapley value does completely eliminate all this issues

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- An similar idea in Noncooperative Game Theory: "The price of anarchy" (Roughgarden & Tardos STOC 2002) measures the efficiency of a system due to selfish behavior of its agents: "the worst" Nash equilibrium versus centralized solution.

An alternative approach!

- Instead of searching for a fair solution: **How fair can be in the worst case a rational solution?**
- An similar idea in Noncooperative Game Theory: "The price of anarchy" (Roughgarden & Tardos STOC 2002) measures the efficiency of a system due to selfish behavior of its agents: "the worst" Nash equilibrium versus centralized solution.
- Cooperative games: The optimum is obtained when all agents cooperate. Challenging problem: cost allocation fairness

In short...

Worst-case fairness in Cooperative Games

- Cooperative games: how to share a joint cost.
- Rational division: noone pays more than what he would pay on its own.
- A rational scheme **may be unfair.**
- What is the most unfair division scheme that is still rational ?

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- Let the vector $x = (x_i)_{i \in N}$, Theil index of x is

$$T(x) = \frac{1}{|N|} \cdot \sum_{i \in N} \frac{x_i}{\overline{x}} \cdot \ln(\frac{x_i}{\overline{x}}),$$

  where $\overline{x} = \frac{x_1 + \ldots + x_{|N|}}{|N|}$. Related with Shannon entropy

$$H(P) = -\sum_{i \in N} p_i \log_2(p_i).$$

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- Cooperative game fairness is defined as

$$p(G) = sup\{T(x) : x \in core(G) \cap \mathbf{Z}^N\}.$$

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FIND the worst case fairness imputation (solution) of such weighted voting game

easy result

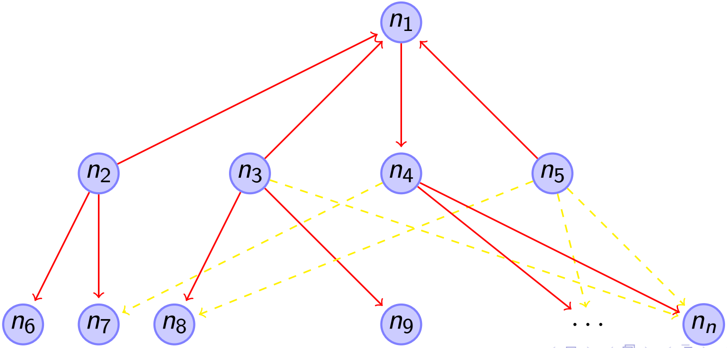
Theorem

Let $\Gamma = ([n], \{w_i\}_{i \in [n]})$ *be a weighted voting game with nonempty core. Without loss of generality assume that players* $1, 2, \ldots K$ *are all the veto players* ($K \geq 1$).

Then the vector $P = (1, 0, \ldots, 0)$ *is a WCF-imputation for the worst case fairness in* Γ.

Example: A Spanning Tree game

- Vertices of a graph G: players, want to form a spanning tree.
- Each player buys some edges it is adjacent to, each edge at unit cost.
- Want: cost sharing solution for (some) spanning tree that minimizes the entropy of the cost distribution.
- THEOREM: NP-hard



Submodularity and Matroids

Matroids

- Common extension of vector spaces and spanning trees.
- universe U, with notion of independent sets of elements.

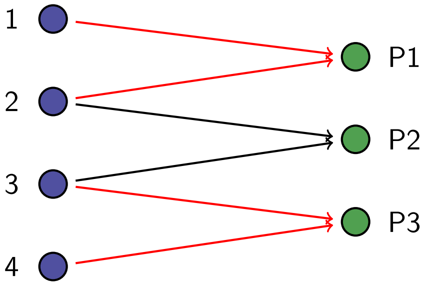
$A \subseteq B, B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$ ("indep. sets closed under subset relation.")

$A, B \in \mathcal{I}, |A| < |B| \Rightarrow (\exists x \in B \setminus A) : A \cup \{x\} \in \mathcal{I}$. ("basis exchange.")

Matroids and submodular functions

Lovász: Every integral polymatroid can be represented using a certain matroid.

Set Cover and matroids



- transversal matroid: "independent sets of edges cover each left node _at most once_".
- cover: basis (each left node covered exactly once).

Minimum Entropy Submodular Set Cover

Definition

GIVEN: Matroid $M = (U, \mathcal{I})$ and sets $P_1, P_2, \ldots, P_m$ covering U.
WANT: Basis B of M and cover $c : B \to \{P_1, P_2, \ldots, P_m\}$ of basis B minimizing the entropy of the cover.

Equivalent Definition (similar to SSC)

GIVEN: Integral polymatroid $f : \mathcal{P}([m]) \to \mathbf{Z}_+$.
WANT: **Modular** function $g : \mathcal{P}([m]) \to \mathbf{Z}_+$ with

$$g([m]) = f([m]) \text{ and } g(S) \leq f(S) \ \forall S \subseteq [m].$$

minimizing $Ent[g]$.

GREEDY algorithm

GREEDY:

INPUT: Integer polymatroid f

Start with $\Gamma_0 = \emptyset$
At stage r
 let $Greedy_r = argmax[f(\Gamma_{r-1} \cup \{i\}) - f(\Gamma_{r-1}]$;
 let $g(Greedy_r) = f(\Gamma_{r-1} \cup \{Greedy_r\}) - f(\Gamma_{r-1})$.

Figure : Greedy algorithm.

Natural question

Does the $log_2(e)$ additive approximation extend to submodularity?

Answer: yes, modulo problem-dependent packing constant.

Main result

THEOREM:

Given instance f of MESSC, GREEDY produces a solution GR

$$Ent(GR) \leq \frac{1}{\alpha} \cdot [Ent(OPT) + \log_2(e)] + (1 - \frac{1}{\alpha}) \log_2(n).$$

where α is a certain (problem-dependent) packing constant

Observations:

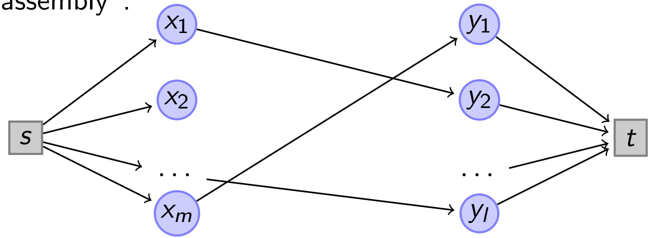
Always $\alpha = \alpha(G) \geq 1$.

When $\alpha(G) = 1$

$$\boxed{Ent[GR] \leq Ent[OPT] + \log_2(e)}$$

Single-level flow constant α

- "We break optimal solution" into parts
- We "reassemble them in the GREEDY solution".
- Reassembly not perfect. α: measures the "quality of the reassembly".



- (x_i) optimal, (y_r) GREEDY. Breaking: subject to capacity constraints.
- A "greedy flow" of full capacity may not exist (it overflows some edge into t and does not fill others).
- Multiplying capacities of edges into t by $\alpha \geq 1$: we enable such flow. $\alpha(G)$: smallest constant.

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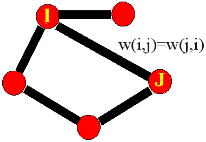
- # Thank you. Questions ?

Application: Induce subgraph game type (I)



- Deng și Papadimitriou (Math. Op. Research 1994).
- Let: $G = (V, E)$ be a graph, the players = the vertex of G.
- Nonnegative weights $w_{i,j}$ on edges: "Increasing the trade volume between i, j when the border taxes are eliminate"
- $S \subseteq V$ subcoalition: $v(S) = \sum_{i,j \in S} w_{i,j}$. The payoff when the taxes are eliminate inside S.

Application: Induce subgraph game type (II)

- We want: the worst division of joint payoff.
- Nontrivial idea: Greedy solution corresponds to an weighted variant of MEO.
- GREEDY algorithm: $Ent[f_{GREEDY}] \leq Ent[f_{OPT}] + \log_2(e)$
- Another algorithm *BIASED*: "the best connected take all by taxe free" $Ent[f_{BIASED}] \leq Ent[f_{OPT}] + \mathbf{1}$