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Heapable subsequences and related concepts

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Introduction

Starting Point: Longest Increasing Sequence

3 2 5 7 1 6 9

- Given n (integer) numbers $a_1, a_2, \ldots, a_n$ find the longest subsequence (not necessarily contiguous) that is increasing
- First-year algorithms: Dynamic programming.
- Another (greedy, also first-year) algorithm: Patience sorting.

Start (greedily) building decreasing piles. When not possible, start new pile.

Patience sorting

16, 25, 18, 2, 4, 35, 3, 7, 32, 9, 20

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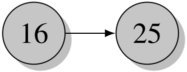

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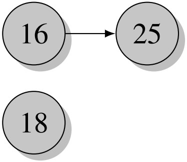

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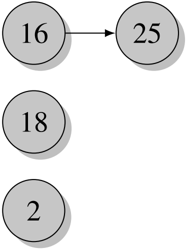

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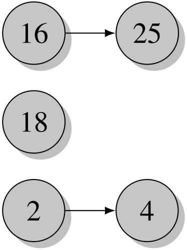

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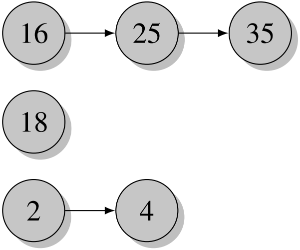

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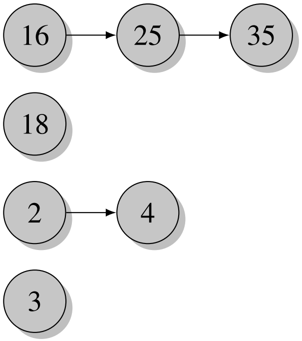

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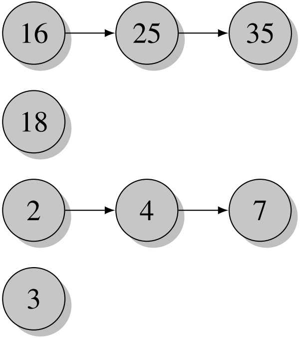

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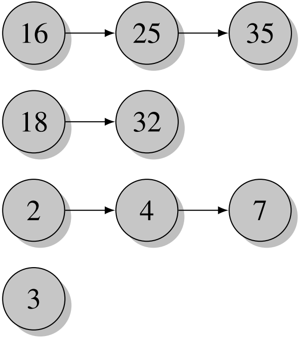

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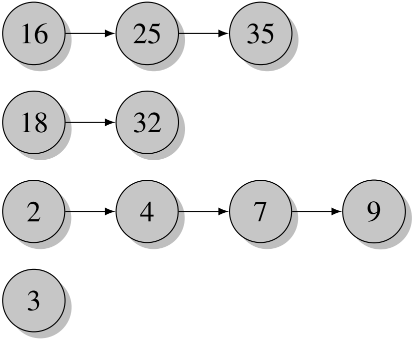

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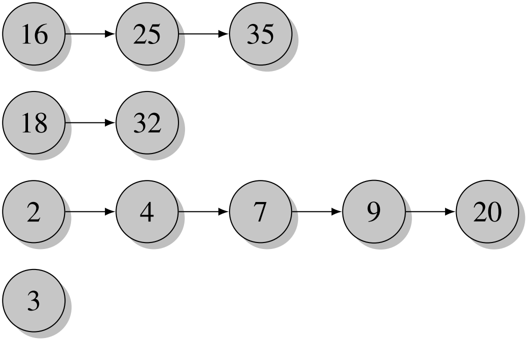

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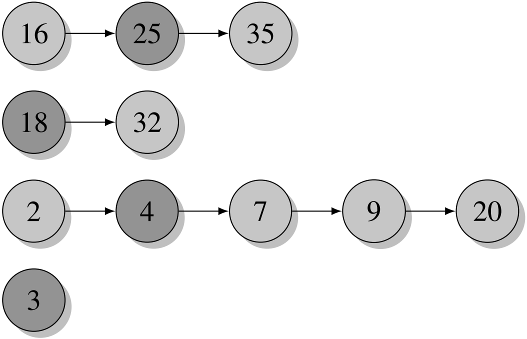
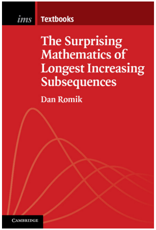


- Partitions the array into increasing (up-)sequences
- SUS(A): the minimal number of such subsequences.
- THEOREM(classical): $SUS(A) = LDS(A)$.

Longest increasing sequence of a random permutation

$$E_{\pi \in S_n}[LIS(\pi)] = 2\sqrt{n} \cdot (1 + o(1)).$$

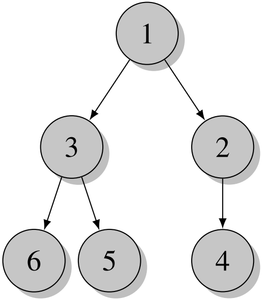
- Logan-Shepp (1977), Veršik-Kerov (1977), Aldous-Diaconis (1995)
- Very rich problem. Connections with nonequilibrium statistical physics and theory of Young tableaux



From data structures to patterns in sequences

Byers, Heeringa, Mitzenmacher, Zervas ANALCO'2011

Sequence of integers A is heapable if it can be inserted into binary heap-ordered tree (not necessarily complete) without having to call HEAPIFY.



Example: 1 3 2 6 5 4   Counterexample: 5 1 . . .

Byers et al. results on heapability

- Polynomial time algorithm for heapability.
- complete heapability: NP-complete
- Longest Heapable Subsequence (LHS): complexity open.
- But with high probability $LHS(\pi) = n - o(n)$, where $\pi \in S_n$ is a random permutation.

Intuition: heapability "weak" versions of increasing sequences.
Recall 1 3 2 6 5 4.
Question (Byers et al.): Does the theory of LIS extend to heapable sequences ?

”Patience heaping”

k-heapable: heapable into a *k*-ary min-heap

$MHS_k(A)$: Extension of SUS. the smallest number of *k*-heapable subsequences in a decomposition of A.

Slot of a node: the k (free) positions where children may grow.

Algorithm 1.1: PATIENCE-HEAPING(W)

INPUT $W = (w_1, w_2, \ldots, w_n)$ a list of integers.
Start with empty heap forest $T = \emptyset$.
for i in range(n):
  if (there exists a slot where X_i can be inserted):
    insert X_i in the slot with the lowest value.
  else :
    start a new heap consisting of X_i only.

Patience heaping

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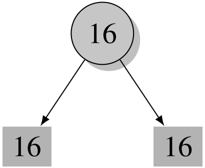

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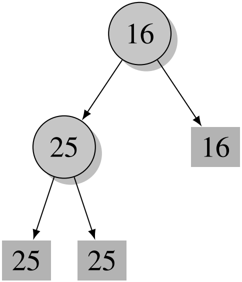

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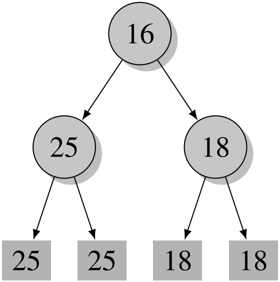

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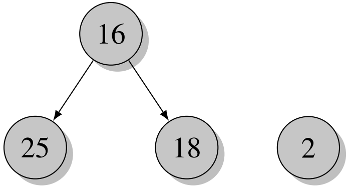

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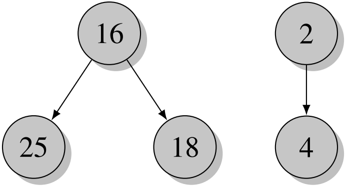

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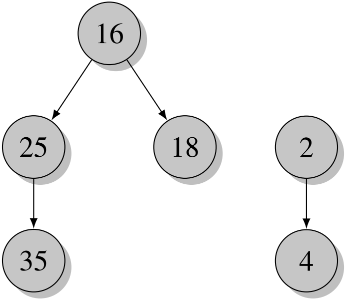

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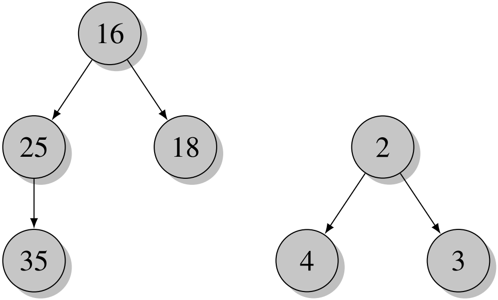

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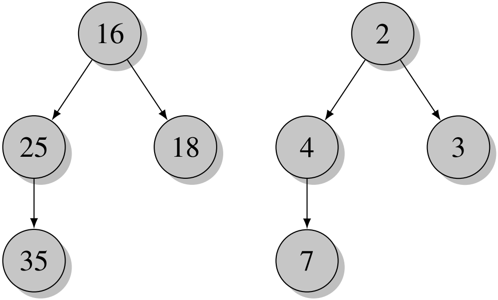

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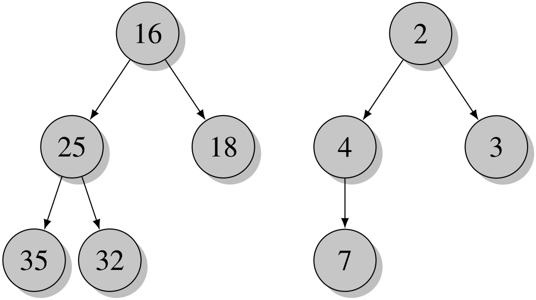

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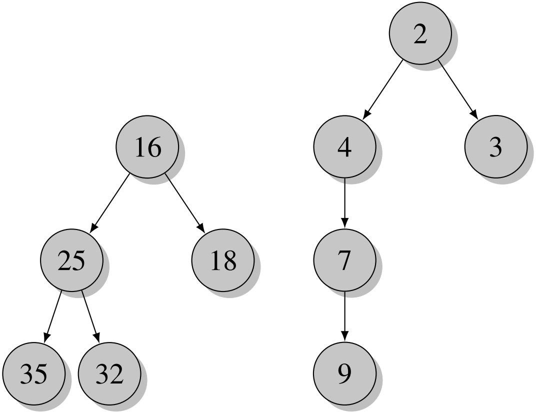

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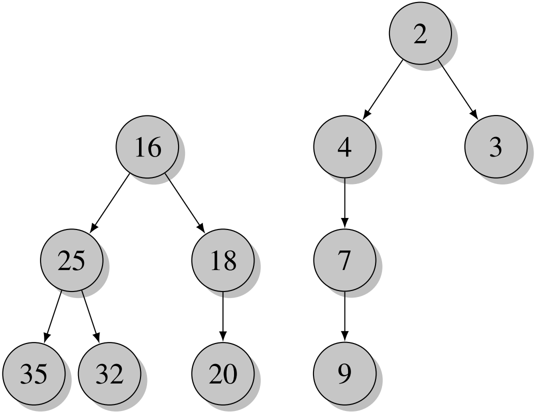


Two (easy) results

THEOREM 1: ”Patience heaping” computes $MHS_k(A)$.
THEOREM 2: (a). there exists sequences X such that
(a). $MHS_k(X) < MHS_{k-1}(X) < \ldots < MHS_1(X)$.
(b). $\sup_X [MHS_{k-1}(X) - MHS_k(X)] = \infty$.

Proof Ideas:

- (a). Define domination relation between multisets of slots.
- Greedy insertion dominates any other insertion+ induction.

  If GREEDY creates new heap then any other algorithm does.

- (b). Thm.1 + Example.

Scaling of MHS_k

How does $E[MHS_k(\pi)]$, where π is a random in S_n, behave ?

- Increasing $\equiv$ "1-heapable". $E[LIS(A)] \sim 2\sqrt{n}$.
- For any k growth at least logarithmic.

THEOREM 3: For every fixed $k, n \geq 1$

$$E_{\pi \in S_n}[MHS_k(\pi)] \geq H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}, \quad (1)$$

the n'th harmonic number, $\sim \ln(n)$.

Proof Idea:
Sequence minima start new heaps.

A beautiful conjecture

CONJECTURE: We have

$$\lim_{n\to\infty} \frac{E[MHS_2[\pi]]}{\ln(n)} = \phi,$$

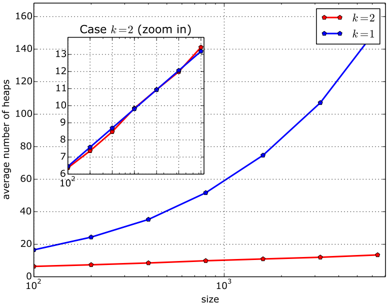
with $\phi = \frac{1+\sqrt{5}}{2}$ the golden ratio.

More generally, for an arbitrary $k \geq 2$ the relevant scaling is

$$\lim_{n\to\infty} \frac{E[MHS_2[\pi]]}{\ln(n)} = \frac{1}{\phi_k}, \tag{2}$$

where ϕ_k is the unique root in $(0, 1)$ of equation $X^k + X^{k-1} + \ldots + X = 1$.

Evidence: Simulations.



- We kind of know what's going on.
- Can make nonrigorous computations that match experimental predictions
- . . . like physicists do !

We just can't make all steps of the argument rigorous !

Hammersley's process: an interacting particle system.

Tops of piles in patience sorting = live particles in Hammersley's process:

- Particles arive at integer times as random real numbers $X_i \in [0, 1]$.
- Particle X_i kills closest live particle X_j, $X_i < X_j$ (if any)

- studied in the area of interacting particle systems, a field between probability theory and (Nonequilibrium) Statistical Physics.
- relative of a more famous process, the so-called Totally Asymmetric Exclusion Process (TASEP)
- Most illuminating proof of $E[LIS(\pi)] \sim 2\sqrt{n}$ (Aldous-Diaconis) analysis of the so-called hydrodynamic limit of Hammersley's process.

Physics of patience heaping ?

Hammersley's process with k lifelines (HAM$_k$):

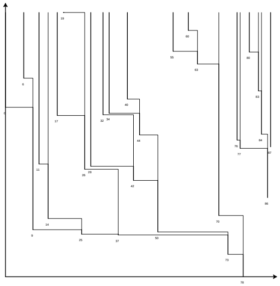
- "Particles" = random numbers $X_i \in [0, 1]$.
- each "particle" is initially endowed with k lives.

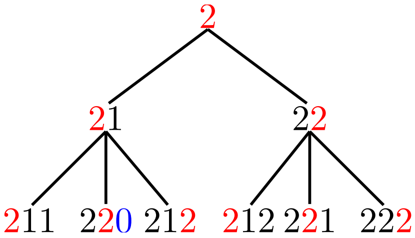
- X_i ~~kills~~ takes one lifeline from closest live X_j, $X_i < X_j$ (if any)

THEOREM 4: At time n the multiset of free slots in patience heaping corresponds (with multiplicities) to live particles in Hammersley's process with k lifelines.

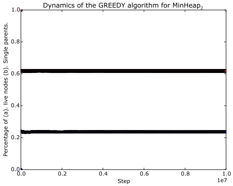
Combinatorial view of process HAM_k

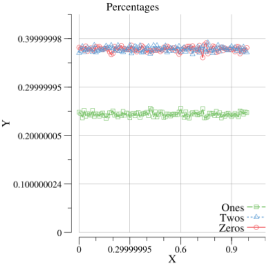
- Words over alphabet $0, 1, 2$ and a (conventional leading) -1.
- Start with $W_0 = -1$ (leftmost marker; not shown below)
- Choose a random position to the right of -1. Put there a 2. Remove 1 from the closest nonzero digit to the right (if any).
- # of heaps = # of insertions that don't remove a lifeline.





A ”physicist” explanation for the dynamics of HAM_k





Exp: 5 indep. runs, 10.000.000 steps. Final vals: Fig 1. First trajectory: Fig 2, once every 10.000 steps. Fig 3: binned densities.

- 1. $n \to \infty$: Limit of W_n = compound Poisson process. W_n = random string of 0,1,2 (densities d_0, d_1, d_2).
- 2. Assuming well mixing of digits one can write evolution equations that predict (constant) limit values of d_0, d_1, d_2.
- 3. From this: ”on average” the probability that the number of heaps increases at stage $n \sim \frac{1+\sqrt{5}}{2 \cdot n}$.

Rigorous result on HAM$_k$

$d_0(n), d_1(n), d_2(n)$: average densities of digits 0,1,2 in word W_n (discarding -1).

FIRST STEP: There exist constants $d_0, d_1, d_2 \in [0, 1]$ such that

$$\lim_{n\to\infty} d_i(n) = d_i, i = 0, 1, 2.$$

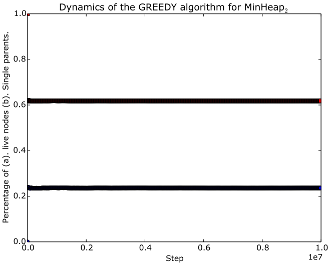
Tool: subadditivity/Fekete's Lemma: "If a_n is a sequence with $a_{m+n} \leq a_m + a_n$ then $\lim_{n\to\infty} a_n/n$ exists."

In the paper: Fekete's lemma applies to some suitable independent linear combinations of $d_i(n)$.

A "physicist" explanation for the dynamics of HAM_k

Empirically $d_1 + d_2 = \frac{\sqrt{5}-1}{2} \sim 0.618\ldots$, $d_2 = \sqrt{5} - 2 \sim 0.236$ and of course $d_0 + d_1 + d_2 = 1$.

Five independent runs, each with 10.000.000 simulation steps. Final values: Figure 1. First trajectory: Figure 2, only once every 10.000 steps (we make 10.000.000 moves !). We've converged very fast.



A physicist like's explanation for the dynamics of HAM_k (III)"

- Assuming well mixing, largest nonzero digit (there are $\sim \frac{\sqrt{5}-1}{2}n$ of them) "has rank $\sim (n - \frac{\sqrt{5}+1}{2})$"
- The probability that the new particle at time n come above this element, therefore increasing the number of heaps, is $\sim \frac{\sqrt{5}+1}{2n}$.
- Thus $E[MHS_2(\pi)]$ is $\sim \frac{\sqrt{5}+1}{2} \cdot H_n \sim \frac{\sqrt{5}+1}{2} \ln(n)$.

Will explain/substantiate all these in a subsequent, "physics-like" paper.