Optimization problems via tree decompositions and potential maximal cliques

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About us

- Université d'Orléans
- Laboratoire d'Informatique Fondamentale d'Orléans (LIFO)
- équipe Graphes, Algorithmes et Modèles de calcul

- graph algorithms for NP-hard optimization problems

How to cope with NP-hard optimization problems?

COLORING, MAX INDEPENDENT SET, TRAVELLING SALESMAN...

- heuristics
- approximation algorithms

- graph classes
- parameterized algorithms: complexity $f(k) \cdot \text{poly}(n)$ for some "small" parameter k
- exact (moderately exponential) algorithms: "best possible" exponential running time

In this talk: tree decompositions, treewidth and so on

- Idea: every graph can be seen as a "generelized tree".
- treewidth of G: measures "how far" is G from the class of trees

G	$\mathrm{tw}(G)$
tree	1
cycle	2
$a \times b$ grid	$\min(a, b)$
complete graph	$n - 1$

Courcelle's theorem: large family of problems[1] can be solved in time $f(\mathrm{tw}) \cdot n$

[1] Expressible in monadic second-order logic.

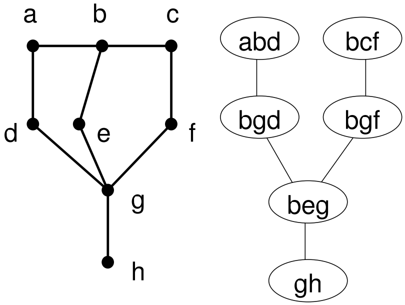
Treewidth

[Robertson, Seymour 84/86]

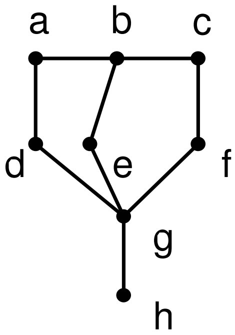
$G \to TG$.

- Each vertex/edge of G is covered by some bag.
- For any vertex x of G, the bags containing x induce a connected subtree of T.

tw$(G) \leq t$ if G has a tree decompositions with bags of size $\leq t+1$.



Minimal separators



Definition

$S \subseteq V$ is a minimal a, b-separator if S separates a and b and it is inclusion-minimal for this property.

Definition

S is a minimal separator of G there exist vertices a, b s.t. S is a minimal a, b-separator.

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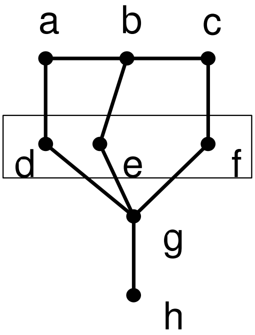

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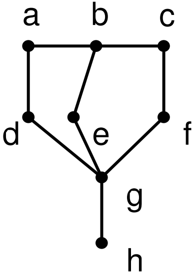
Decomposing with minimal separators



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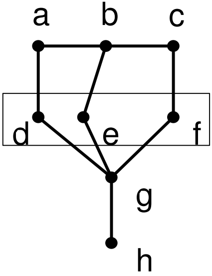

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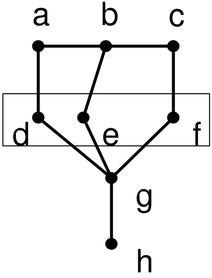

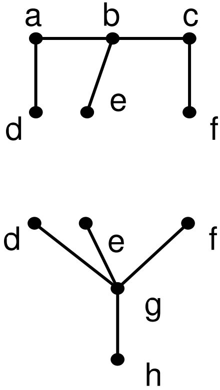

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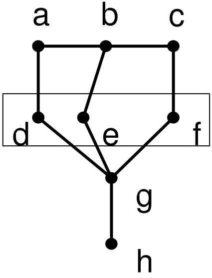

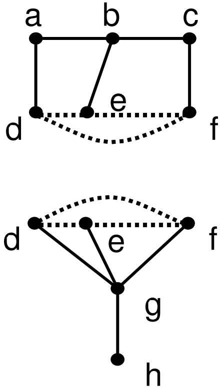

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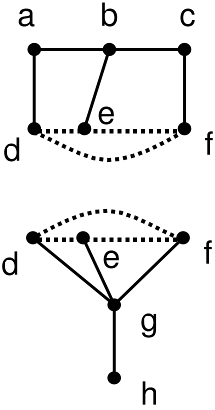

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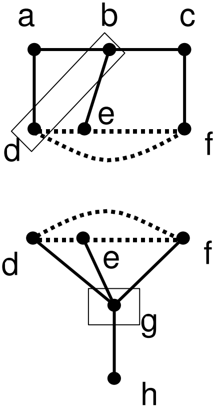

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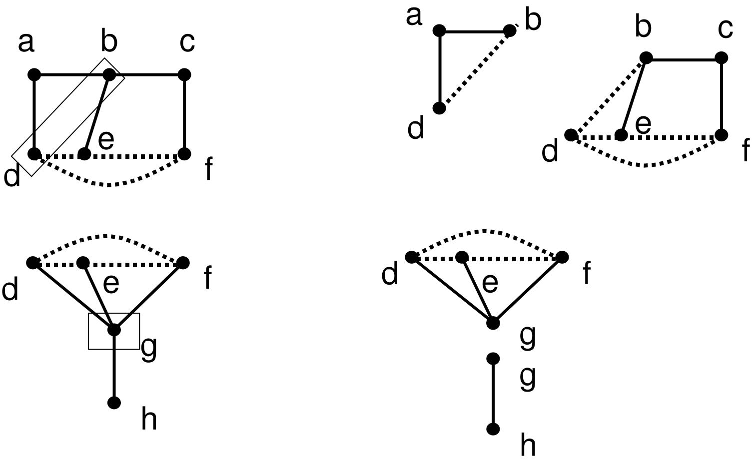

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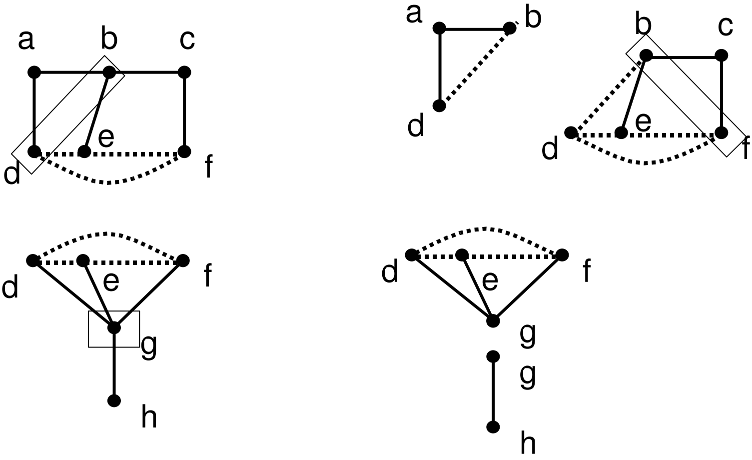

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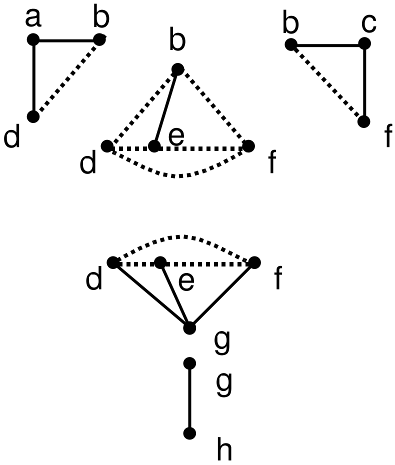

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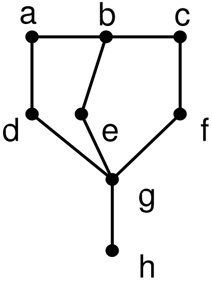

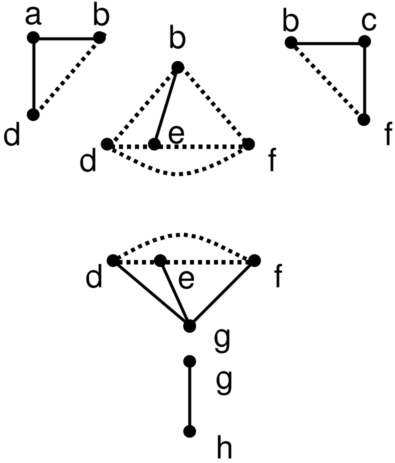

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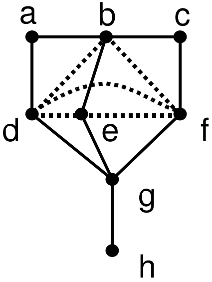

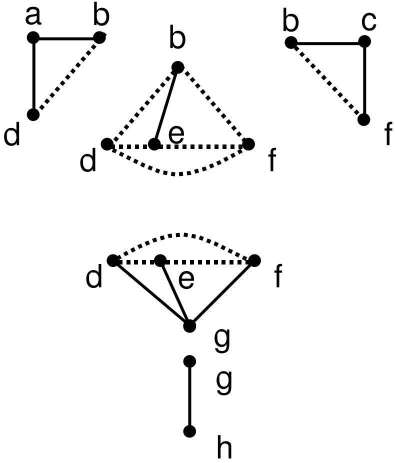

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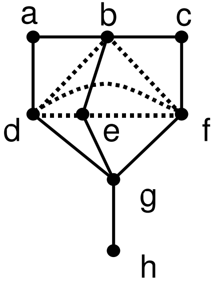
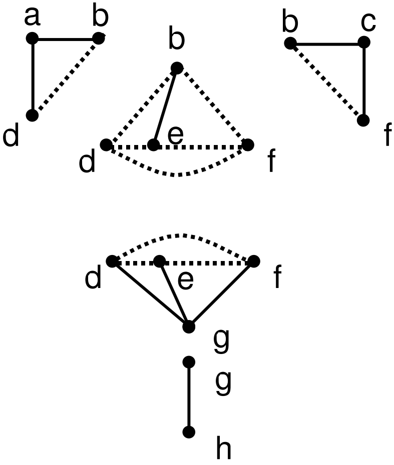




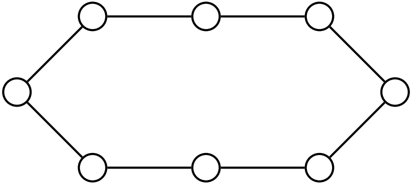
Theorem ([Parra, Schaeffler 97])

Decomposing through minimal separators → minimal tree decompositions.

Potential maximal cliques

Definition ([Bouchitté, T. 1999])

A set of vertices Ω is a potential maximal clique of G if there is a minimal tree decomposition TG of G such that Ω is a bag in TG.



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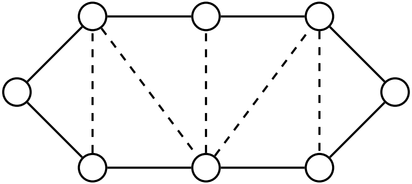

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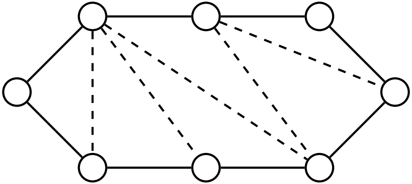

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Applications of potential maximal cliques

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Given as input a graph G and the set of its potential maximal cliques, we can solve the following problems in time $\mathcal{O}(\sharp\,\mathrm{p.\,m.\,c.} \cdot \mathrm{poly}(n))$*:*

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- MAXIMUM INDEPENDENT SET, MAXIMUM INDUCED FOREST, LONGEST INDUCED PATH...
- ...MAXIMUM INDUCED SUBGRAPH OF $\mathrm{tw} \leq t$ SATISFYING $\mathcal{P}$ *[Fomin, T., Villanger, SODA 2014 & SIAM J. Comp 2015]*

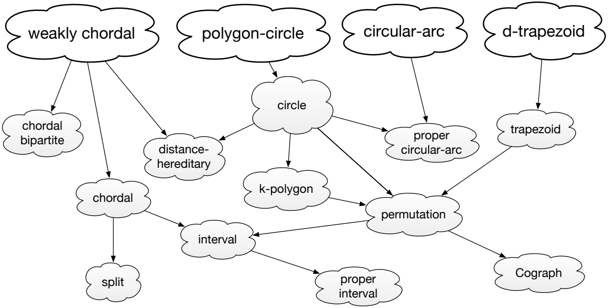
Further consequences

All these problems (TREEWIDTH, MAXIMUM INDEPENDENT SET, FEEDBACK VERTEX SET... can be solved in time

- poly(n) for all graph classes with polynomially many minimal separators,
- $O(1.737^n)$ for arbitrary graphs [Fomin, T., Villanger 2015],
- $4^{\text{vertex\_cover}} \cdot \text{poly}(n)$ [Fomin, Liedloff, Montealegre, T. 2014]
- FPT for other parameters [Liedloff, Montealegre, T. 2015]

To wrap up

Potential maximal cliques can be used to obtain algorithmic meta-theorems for graph classes, exact (exponential) algorithms, parameterized algorithms.



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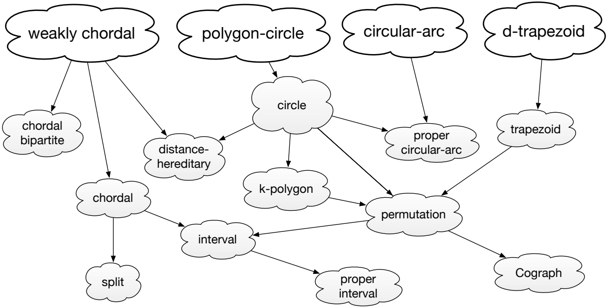

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Vă mulțumesc! Întrebări?